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The $\bar{\partial}$-approach to monochromatic inverse scattering in three dimensions

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Abstract

We discuss a method for monochromatic inverse scattering in three dimensions of [R.Novikov 2005] and implemented numerically in [Alekseenko, Burov, Rumyantseva 2007]. This method is obtained as a development of the $\bar{\partial}$-approach to inverse scattering at fixed energy in dimension $d \geq 3$ of [Beals, Coifman 1985] and [Henkin, R.Novikov 1987] and involves, in particular, some results of [Faddeev 1965, 1974] and some ideas of the soliton theory (in particular, some ideas going back to [Manakov 1976] and [Dubrovin, Krichever, S.Novikov 1976]). Our studies go back also, in particular, to [Regge 1959].

1. Introduction

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{1.1}$$

where

$$\begin{aligned} &v \ \text{ is a sufficiently regular function on } \mathbb{R}^d \\ &\text{with sufficient decay at infinity.} \end{aligned} \tag{1.2}$$

For equation (1.1) we consider the scattering amplitude $f(k,l)$, where $(k,l) \in \mathcal{M}_E$,

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : \quad k^2 = l^2 = E\}, \quad E > 0. \tag{1.3}$$

For definitions of the scattering amplitude see formula (1.4) below and, for example, [F3], [FM]. The scattering amplitude f arises, in particular, as a coefficient with scattered spherical wave $e^{i|k||x|}/|x|^{(d-1)/2}$ in the asymptotics of the wave solutions $\psi^+(x,k)$ describing scattering of incident plan waves e^{ikx} for equation (1.1):

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f\big(k, |k|\frac{x}{|x|}\big) + \\ &o\big(\frac{1}{|x|^{(d-1)/2}}\big) \ \text{ as } \ |x| \to +\infty, \end{aligned} \tag{1.4}$$

where $k \in \mathbb{R}^d$, $k^2 = E$, $c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}$.

Given v, to determine f one can use, in particular, the following integral equation

$$f(k,l) = \hat{v}(k-l) - \int\limits_{\mathbb{R}^d} \frac{\hat{v}(m-l)f(k,m)dm}{m^2 - k^2 - i0}, \tag{1.5}$$

where

$$\hat{v}(p) = \big(\frac{1}{2\pi}\big)^d \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{1.6}$$

In addition, (1.5) is an equation for $f(k, \cdot)$ on $\mathbb{R}^d$ for each fixed $k \in \mathbb{R}^d$, $k^2 = E$, and the scattering amplitude arises as the restriction $f\big|_{\mathcal{M}_E}$.

We consider the following monochromatic inverse scattering problem for equation (1.1):

Problem 1.1 Given f on $\mathcal{M}_E$ at fixed $E > 0$, find v on $\mathbb{R}^d$ (at least approximately, but sufficiently stably for numerical implementations).

Note that the Schrödinger equation (1.1) at fixed positive energy can be considered also as the acoustic equation at fixed frequency (see, for example, Section 5.2 of [HN] and Section 4 of the present article).

Therefore, Problem 1.1 is also a basic problem of the monochromatic ultrasonic tomography. Actually, the creation of effective reconstruction methods for inverse scattering in multidimensions (and especially in three dimensions) was formulated as a very important problem many times in the mathematical literature, see, for example, [Gel], [F3], [Gro]. In particular, as it is mentioned in [Gro]: "For example, an efficient inverse scattering algorithm would revolutionize medical diagnostic, making ultrasonic devices at least as efficient as current X-ray analysis". The works [No8], [ABR3] can be considered as recent steps to this ojective.

In [No8] we assume that

$$v \in W^{n,1}_s(\mathbb{R}^d) \ \text{ for some } \ n \in \mathbb{N},\ n > d-2, \ \text{ and some } \ s > 0, \tag{1.7}$$

where

$$W^{n,1}_s(\mathbb{R}^d) = \{u: \ \Lambda^s \partial^J u \in L^1(\mathbb{R}^d) \ \text{ for } \ |J| \le n\}, \tag{1.8a}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^d J_i, \quad \partial^J u(x) = \frac{\partial^{|J|} u(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}},$$
$$\Lambda^s w(x) = (1 + |x|^2)^{s/2} w(x), \quad x \in \mathbb{R}^d.$$

Suppose, first, that

$$\|v\|^{n,1}_s = \max_{|J| \le n} \|\Lambda^s \partial^J v\|_{L^1(\mathbb{R}^d)} \tag{1.8b}$$

is so small for fixed n, s, d and $E_0 > 0$ that the following well-known Born approximation

$$f(k,l) \approx \hat{v}(k-l), \quad (k,l) \in \mathcal{M}_E, \quad E \ge E_0, \tag{1.9}$$

where f is the scattering amplitude for equation (1.1) and $\hat{v}$ is the Fourier transform of v (see (1.5),(1.6)), is completely satisfactory. Then Problem 1.1 (for fixed $d \ge 2$ and fixed $E \ge E_0$) is reduced to finding v from $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$, where

$$\mathcal{B}_r = \{p \in \mathbb{R}^d: \quad |p| < r\}, \quad r > 0. \tag{1.10}$$

This linearized monochromatic inverse scattering problem can be solved by the formula

$$v(x) = v^{lin}_{appr}(x,E) + v^{lin}_{err}(x,E),$$
$$v^{lin}_{appr}(x,E) = \int\limits_{\mathcal{B}_{2\sqrt{E}}} e^{-ipx}\hat v(p)dp, \quad v^{lin}_{err}(x,E) = \int\limits_{\mathbb{R}^d\setminus\mathcal{B}_{2\sqrt{E}}} e^{-ipx}\hat v(p)dp, \tag{1.11}$$

$x\in\mathbb{R}^d$, $E\ge E_0$. In addition, if $v\in W^{n,1}_0(\mathbb{R}^d)$ for some $n>d$ and $\|v\|^{n,1}_0\le C$ (where we use definitions (1.8a), (1.8b)), then

$$|\hat v(p)| \le c_1(n,d)C(1+|p|)^{-n}, \quad p\in\mathbb{R}^d, \tag{1.12a}$$

and, therefore,

$$|v^{lin}_{err}(x,E)| \le c_2(n,d)CE^{-(n-d)/2}, \quad x\in\mathbb{R}^d, \quad E\ge E_0, \tag{1.12b}$$

where $c_1(n,d)$, $c_2(n,d)$ are some positive constants.

Thus, in the Born approximation (1.9) (that is in the linear approximation near zero potential) we have that:

(1) f on $\mathcal{M}_E$ stably determines $v^{lin}_{appr}(x,E)$ of (1.11) and

(2) the error $v^{lin}_{err}(x,E) = v(x) - v^{lin}_{appr}(x,E) = O(E^{-(n-d)/2})$ in the uniform norm as $E\to+\infty$ for n-times smooth v in the sense (1.7), $n>d$. (In particular, $v^{lin}_{err} = O(E^{-\infty})$ in the uniform norm as $E\to+\infty$ for v of the Schwartz class on $\mathbb{R}^d$.)

For general nonlinearized case for $d=3$ analogs of (1.9), (1.11), (1.12b) are given in [No8] (and implemented numerically in [ABR3]). These results are presented below in Sections 2, 3 and 4. However, before this some additional remarks may be in order.

Remark 1.1. One can see that $v^{lin}_{appr}(x,E)$ at fixed $E>0$ is not a precise reconstruction of v, in general. On the other hand, if v satisfies (1.7) and, in addition, is compactly supported or exponentially decaying at infinity, then $\hat v$ on $\mathcal{B}_{2\sqrt{E}}$ uniquely determines $\hat v$ on $\mathbb{R}^d\setminus\mathcal{B}_{2\sqrt{E}}$ (at fixed $E>0$) by an analytic continuation and, therefore, in the Born approximation (1.9) (for $d\ge 2$) f on $\mathcal{M}_E$ (at fixed $E\ge E_0$) uniquely determines v on $\mathbb{R}^d$. However, in contrast with v^{lin}_{appr} the latter determination is not sufficiently stable for direct numerical implementation.

Remark 1.2. The works [No2], [No3], [No4], [No5] give, in particular, uniqueness theorems and precise reconstructions for the problem of finding v on $\mathbb{R}^d$ from f on $\mathcal{M}_E$ (at fixed $E>0$ for $d\ge 2$) for the general nonlinearized case. These results are global for $d\ge 3$ (and assume that v is sufficiently small in comparison with fixed $E>0$ for $d=2$). However, these reconstructions of [No2], [No3], [No4], [No5] are not sufficiently stable for direct numerical implementation (because of the nature of Problem 1.1 explained already for the linearized case (1.9) in Remark 1.1). Actually, any precise reconstruction of v, where $v\in C^n(\mathbb{R}^d)$ and $supp\, v\in\mathcal{B}_r$ for fixed $n\in\mathbb{N}$ and $r>0$, from f on $\mathcal{M}_E$ at fixed $E>0$ is exponentially unstable (see [Mand]). In the present text (as well as in [No6], [No7], [No8]) we do not discuss such reconstructions in detail.

Remark 1.3. In [HN] it is shown, in particular, that if $\Lambda^s v \in L^\infty(\mathbb{R}^d, \mathbb{R})$ for some $s > d$, where Λ is the weight of (1.8), and if $d \geq 2$, then for any fixed E and δ, where $0 < \delta < E$, the scattering amplitude f on $\cup_{\lambda\in[E-\delta,E+\delta]} \mathcal{M}_\lambda$ uniquely determines the Fourier transform $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$. However, unfortunately, this determination of [HN] involves an analytical continuation and, therefore, is not sufficiently stable for direct numerical implementation.

Remark 1.4. In [Ch] an efficient numerical algorithm for the reconstruction from multi-frequency scattering data was proposed in two dimensions, but [Ch] gives no rigorous mathematical theorem.

Remark 1.5. On the other hand (in comparison with results mentioned in Remarks 1.2, 1.3, 1.4) in [No6], [No7] we succeeded to give stable approximate solutions of nonlinearized Problem 1.1 for $d = 2$ and v satisfying (1.7), $n > d = 2$, with the same decay rate of the error terms for $E \to +\infty$ as in the linearized case (1.9), (1.11), (1.12b) (or, more precisely, with the error terms decaying as $O(E^{-(n-2)/2})$ in the uniform norm as $E \to +\infty$). In [No6], [No7] we proceed from the fixed-energy inverse scattering reconstruction procedure developed in [No1], [GM], [No2], [No4] for $d = 2$. In turn, the works [No1], [GM], [No2], [No4] are based on some ideas and approaches of the solition theory (in particular, on some ideas and approaches going back to [M1], [DKN], [M2], [ABF], [NV1], [NV2]) and on some results of [F1], [F2], [F3], [GN]. The work [No6] was stimulated also by the "one-dimensional" works [MNPSF] and [HeNo]. The reconstruction procedure of [No1], [GM], [No2], [No4] was implemented numerically in [BMRSVZ], [BBMRS], [BMR].

2. Reconstruction of [No8]

2.1. Preliminary presentation. In [No8] for v satisfying (1.7), $n > d$, for general nonlinearized case for $d = 3$ we succeeded, in particular, to give a stable reconstruction

$$f \ \text{ on } \ \mathcal{M}_E \xrightarrow{\textit{stable reconstruction}} v^{\pm}_{appr}(\cdot, \tau, E) \ \text{ on } \ \mathbb{R}^3 \tag{2.1}$$

such that

$$\begin{gathered} v(x) - v^{\pm}_{appr}(x, \tau, E) = O(E^{-(n-3)/2} \ln E) \\ \text{in the uniform form as } \ E \to +\infty, \ x \in \mathbb{R}^3, \end{gathered} \tag{2.2}$$

where $\pm$ and τ are fixed parameters. In addition, $0 < \tau < \delta(s, n, C)$ for appropriate δ (see formula (2.8) and conditions (2.6) discussed below), where $C \geq \|v\|^{n,1}_s$. That is in [No8] we succeded, in particular, to give a stable approximate solution of nonlinearized Problem 1.1 for $d = 3$ and v satisfying (1.7), $n > d = 3$ with the error term $v(\cdot) - v^{\pm}_{appr}(\cdot, \tau, E)$ decaying with almost the same rate for $E \to +\infty$ as in the linearized case (1.9), (1.11), (1.12b).

Reconstruction (2.1) (with estimate (2.2)) is based, in turn, on the following reconstruction of [No8] for v satisfying (1.7), $n > d = 3$,

$$f \ \text{ on } \ \mathcal{M}_E \xrightarrow{\textit{stable reconstruction}} \hat{v}^{\pm}_{appr}(\cdot, \tau, E) \ \text{ on } \ \mathcal{B}_{2\tau\sqrt{E}} \tag{2.3}$$

such that

$$|\hat v(p)-\hat v^{\pm}_{appr}(p,\tau,E)|=\frac{O(E^{-(n-\mu_0)/2})}{(1+|p|)^{\mu_0}},\ \ \text{as}\ \ E\to+\infty,\ p\in\mathcal{B}_{2\tau\sqrt{E}},\ \mu_0\in[2,n[, \tag{2.4}$$

where $\hat v$ is (the Fourier transform of v) defined by (1.6), $O(E^{-(n-\mu_0)/2})$ is independent of p, and μ_0, $\pm$, τ are fixed parameters. In addition,

$$0<\tau<\delta_1(s,n,\mu_0,C),\quad C\ge\|v\|_s^{n,1}, \tag{2.5}$$

for appropriate δ_1, where, in particular,

$$0<\delta_1<1, \tag{2.6a}$$
$$\delta_1\to 0\ \ \text{as}\ \ C\to+\infty. \tag{2.6b}$$

Using (2.3)-(2.6a) one can see that in (2.3) we even do not try to reconstruct $\hat v$ on $\mathbb{R}^3\backslash\mathcal{B}_{2\sqrt{E}}$. In addition, we think that condition (2.6b) is not necessary and is related mainly with technical details of our proof of [No8].

In terms of the approximate reconstruction $\hat v^{\pm}_{appr}$ of (2.3), (2.4) the approximate reconstruction $v^{\pm}_{appr}$ of (2.1), (2.2) is given by

$$v^{\pm}_{appr}(x,\tau,E)=\int\limits_{\mathcal{B}_{2\sqrt{E}}} e^{-ipx}\hat v^{\pm}_{appr}(p,\tau,E)dp,\quad x\in\mathbb{R}^3, \tag{2.7}$$

where we use (2.3), (2.4) for $\mu_0=3$. In addition,

$$\delta(s,n,C)=\delta_1(s,n,3,C), \tag{2.8}$$

where δ was mentioned in connection with (2.1), (2.2).

A stability estimate for (2.3) is given by formula (3.7) below. More detailed version of (2.4) is given by formula (3.6) below.

Note that before [No6], [No7] for $d=2$ and [No8] for $d=3$, even for real v of the Schwartz class on $\mathbb{R}^d$, $d\ge 2$, no result was given, in general, in the literature on finding v on $\mathbb{R}^d$ from f on $\mathcal{M}_E$ with the error term decaying more rapidly than $O(E^{-1/2})$ in the uniform norm as $E\to+\infty$ (see related discussion given in [No6]).

Note that inverse scattering at fixed energy for the Schrödinger equation (1.1) with spherically symmetric potential v in three dimensions was studied in many works, see [R], [Ne], [CS] and references therein. Nevertheless, the results (of [No8]) consisting in reconstruction (2.1)-(2.4) are new even for spherically symmetric potentials.

Note that the method of [No8] is obtained as a development of the $\bar\partial$-approach to inverse scattering at fixed energy in dimension $d\ge 3$ of [BC] and [HN] and involves, in particular, some results of [F1], [F2], [F3] and some ideas of the "two-dimensional" works mentioned in Remark 1.5. In addition, there is an interesting similarity between [R], on one hand, and [BC], [HN], [No8], on the other hand, consisting in an essential use of the complex analysis.

2.2. Background results. Reconstruction (2.3) is based on properties of the Faddeev function $H(k,p)$, where $k \in \mathbb{C}^3\backslash\mathbb{R}^3$, $p \in \mathbb{R}^3$. For definitions of H see [F3], [HN]. Given v, to determine H one can use, in particular, the following integral equation

$$H(k,p) = \hat{v}(p) - \int\limits_{\mathbb{R}^3} \frac{\hat{v}(p+\xi)H(k,-\xi)d\xi}{\xi^2 + 2k\xi}, \quad k \in \mathbb{C}^3\backslash\mathbb{R}^3,\ p \in \mathbb{R}^3, \tag{2.9}$$

where $\hat{v}$ is (the Fourier transform of v) defined by (1.6). Actually, (2.9) is an equation for $H(k,\cdot)$ on $\mathbb{R}^3$ for each fixed $k \in \mathbb{C}^3\backslash\mathbb{R}^3$. Reconstruction (2.3) is based, in particular, on properties of H on $\Omega_E\backslash Re\,\Omega_E$, where

$$\begin{aligned} &\Omega_E = \{k \in \mathbb{C}^3,\ p \in \mathbb{R}^3 :\ \ p^2 = 2kp,\ \ k^2 = E\},\\ &Re\,\Omega_E = \{k \in \mathbb{R}^3,\ p \in \mathbb{R}^3 :\ \ p^2 = 2kp,\ \ k^2 = E\}, \end{aligned} \tag{2.10}$$

and on properties of

$$H_\gamma(k,p) = H(k + i0\gamma, p), \quad (k,p) \in Re\,\Omega_E, \quad \gamma \in \mathbb{S}^2; \tag{2.11}$$

see formulas (2.20)-(2.29) given below.

To deal with f and H of (1.5) and (2.9) (in the framework of the aforementioned results) we consider, in particular, the functional spaces $C^{\alpha,\mu}(\mathbb{R}^3)$, $\mathcal{H}_{\alpha,\mu}(\mathbb{R}^3)$, $C^\alpha(\mathcal{M}_E)$:

$$C^{\alpha,\mu}(\mathbb{R}^3) = \{u \in C(\mathbb{R}^3) :\ \|u\|_{\alpha,\mu} < +\infty\}, \quad \alpha \in]0,1[,\ \mu \in \mathbb{R}, \tag{2.12}$$

where

$$\|u\|_{\alpha,\mu} = \|\Lambda^\mu u\|_\alpha, \tag{2.13a}$$

$$\Lambda^\mu u(p) = (1 + |p|^2)^{\mu/2} u(p), \quad p \in \mathbb{R}^3, \tag{2.13b}$$

$$\|w\|_\alpha = \sup_{p,\xi \in \mathbb{R}^3,\ |\xi| \le 1} (|w(p)| + |\xi|^{-\alpha}|w(p+\xi) - w(p)|); \tag{2.13c}$$

$$\mathcal{H}_{\alpha,\mu} \ \text{ is the closure of } \ C_0^\infty(\mathbb{R}^3) \ \text{ in } \ \|\cdot\|_{\alpha,\mu},\ \alpha \in]0,1[,\ \mu \in \mathbb{R}, \tag{2.14}$$

where $C_0^\infty(\mathbb{R}^3)$ is the space of infinitely smooth functions on $\mathbb{R}^3$ with compact support;

$$C^\alpha(\mathcal{M}_E) = \{u \in C(\mathcal{M}_E) :\ \|u\|_{C^\alpha(\mathcal{M}_E),0} < +\infty\},\ \alpha \in [0,1[,\ E > 0, \tag{2.15}$$

where

$$\begin{aligned} &\|u\|_{C^0(\mathcal{M}_E),\mu} = \|u\|_{C(\mathcal{M}_E),\mu} =\\ &\sup_{(k,l) \in \mathcal{M}_E} (1 + |k - l|^2)^{\mu/2}|u(k,l)|,\ \mu \ge 0, \end{aligned} \tag{2.16a}$$

$$\begin{aligned} &\|u\|_{C^\alpha(\mathcal{M}_E),\mu} = \max(\|u\|_{C(\mathcal{M}_E),\mu}, \|u\|'_{C^\alpha(\mathcal{M}_E),\mu}),\\ &\|u\|'_{C^\alpha(\mathcal{M}_E),\mu} = \sup_{\substack{(k,l),(k',l') \in \mathcal{M}_E\\ |k-k'| \le 1, |l-l'| \le 1}} (1 + |k - l|^2)^{\mu/2}(|k - k'|^\alpha + |l - l'|^\alpha)^{-1} \times\\ &|u(k,l) - u(k',l')| \ \text{ for } \ \alpha \in]0,1[,\ \mu \ge 0. \end{aligned} \tag{2.16b}$$

We use that if v satisfies (1.7), $d = 3$, then

$$\begin{aligned}&\hat{v} \in \mathcal{H}_{\alpha,n}(\mathbb{R}^3) \ \text{ for } \ \alpha \in \begin{cases}]0, s] & \text{for } \ s < 1 \\ ]0, 1[& \text{for } \ s \geq 1 \end{cases}, \\ &\|\hat{v}\|_{\alpha,n} \leq Const_0(s, \alpha, n)\|v\|_s^{n,1},\end{aligned} \tag{2.17}$$

where $\hat{v}$ is the Fourier transform of v. Further, for obtaining (2.3), (2.4) we use, in particular, that if

$$\hat{v} \in \mathcal{H}_{\alpha,n}(\mathbb{R}^3) \ \text{ for some } \ \alpha \in]0, 1[\ \text{ and some real } \ \mu > 1 \tag{2.18}$$

and

$$\begin{aligned}&\|\hat{v}\|_{\alpha,\mu} \leq D < \frac{E^{\sigma/2}}{Const_1(\alpha, \mu, \sigma)} \\ &\text{for some } \ \sigma \in]0, \min{(1, \mu - 1)}[\ \text{ and some } \ E \geq 1,\end{aligned} \tag{2.19}$$

then (see [No8]):

I. For $k^2 = E$ equations (1.5) for f and (2.9) for H are uniquely solvable in $C^{\alpha,\mu}(\mathbb{R}^3)$ for each fixed k (considered as mentioned in these equations).

II. The function H_γ of (2.11) at fixed E is well-defined and is related with the scattering amplitude f on $\mathcal{M}_E$ by the following equation (of [F2], [F3])

$$\begin{aligned}&h_\gamma(k, l) = f(k, l) + \frac{\pi i}{\sqrt{E}} \int\limits_{\mathbb{S}^2_{\sqrt{E}}} h_\gamma(k, m)\chi((m - k)\gamma)f(m, l)dm, \\ &(k, l) \in \mathcal{M}_E, \quad \gamma \in \mathbb{S}^2 = \mathbb{S}^2_1,\end{aligned} \tag{2.20}$$

where

$$S^2_r = \{m \in \mathbb{R}^3 : \ \ |m| = r\}, \quad r > 0, \tag{2.21}$$

$$\chi(s) = 0 \ \text{ for } \ s \leq 0, \quad \chi(s) = 1 \ \text{ for } \ s > 0, \tag{2.22}$$

and by the formula

$$H_\gamma(k, p) = h_\gamma(k, k - p), \quad (k, p) \in Re\,\Omega_E, \quad \gamma \in \mathbb{S}^2 = \mathbb{S}^2_1, \tag{2.23}$$

where

$$(k, p) \in Re\,\Omega_E \Longleftrightarrow (k, k - p) \in \mathcal{M}_E. \tag{2.24}$$

III. The following $\bar{\partial}$-equation (of [BC], [HN]) holds:

$$\begin{aligned}&\partial_{\bar{k}} H(k, p) = -2\pi \int\limits_{\mathbb{R}^3} (\sum_{j=1}^{3} \xi_j d\bar{k}_j) H(k, -\xi) H(K + \xi, p + \xi) \times \\ &\delta(\xi^2 + 2k\xi) d\xi \ \text{ for } \ H \ \text{ on } \ \Omega_E \backslash Re\,\Omega_E,\end{aligned} \tag{2.25}$$

where δ is the Dirac function:

$$\int\limits_{\mathbb{R}^3} u(\xi)\delta(\xi^2+2k\xi)d\xi = \int\limits_{\{\xi\in\mathbb{R}^3:\ \xi^2+2k\xi=0\}} \frac{u(\xi)}{|J(k,\xi)|}|d\xi_3|, \tag{2.26}$$

where $J(k,\xi) = 4[(\xi_1 + Re\,k_1)Im\,k_2 - (\xi_2 + Re\,k_2)Im\,k_1]$ is the Jacobian of the map $(\xi_1,\xi_2,\xi_3) \to (\xi^2 + 2Re\,k\xi, 2Im\,k\xi, \xi_3)$ and u is a test function. In addition,

$$(k,p)\in\Omega_E \Longrightarrow (k,-\xi)\in\Omega_E, (k+\xi,p+\xi)\in\Omega_E \ \text{ if } \ \xi\in\mathbb{R}^3,\ \xi^2+2k\xi=0. \tag{2.27}$$

IV. The following formula (of [HN]) holds:

$$\hat v(p) = \lim_{(k,p)\in\Omega_E,|k|\to\infty} H(k,p) \ \text{ for each } \ p\in\mathbb{R}^3, \tag{2.28}$$

where $|k| = (|Re\,k|^2 + |Im\,k|^2)^{1/2}$.

V. The following estimates hold:

$$|H(k,p)| \le \frac{D}{(1-E^{-\sigma/2}Const_1(\alpha,\mu,\sigma)D)(1+|p|^2)^{\mu/2}},\ (k,p)\in\Omega_E\backslash Re\,\Omega_E, \tag{2.29}$$

$$\|f\|_{C(\mathcal{M}_E),\mu} \le \frac{D}{1-E^{-\sigma/2}Const_1(\alpha,\mu,\sigma)D}, \tag{2.30a}$$

$$\|f\|_{C^\alpha(\mathcal{M}_E),\mu} \le \frac{Const_2 D}{1-E^{-\sigma/2}Const_1(\alpha,\mu,\sigma)D}, \tag{2.30b}$$

where D and $Const_1$ are the constants of (2.18), see [No8] (and [ER] in connection with (2.30)).

2.3. Reconstruction scheme. Reconstruction (2.3) is based on properties (2.11), (2.20), (2.23), (2.25), (2.28), (2.29), (2.30) of the functions H and f and consists of the following parts:

$$\begin{gathered} f \ \text{ on } \ \mathcal{M}_E \xrightarrow{(2.20),(2.23),(2.30)} H_\gamma(k,p),\ (k,p)\in Re\,\Omega_E,\ \gamma\in\mathbb{S}^2,\ \gamma k=0, \\ \longrightarrow H_\pm(k,p) = H_{\gamma^\pm(k,p)}(k,p), \\ \gamma^\pm(k,p) = \frac{\pm p\times(k-p/2)}{|p||k-p/2|},\ (k,p)\in Re\,\Omega_E,\ |p|\ne 0, \end{gathered} \tag{2.31a}$$

where $\times$ denotes vector product;

$$\begin{gathered} H_+ \ \text{ and } \ H_- \ \text{ on } \ Re\,\Omega_E^\tau \xrightarrow{(2.11),(2.25),(2.29)} H^{appr}_{E,\tau} \ \text{ on } \ \Omega_E^\tau\backslash Re\,\Omega_E^\tau, \\ \Omega_E^\tau = \{(k,p)\in\Omega_E:\ p\in\mathcal{B}_{2\tau\sqrt{E}}\},\ Re\,\Omega_E^\tau = \{(k,p)\in Re\,\Omega_E:\ p\in\mathcal{B}_{2\tau\sqrt{E}}\}, \end{gathered} \tag{2.31b}$$

where τ is the parameter of (2.3), $H^{appr}_{E,\tau}$ is an approximation to H on $\Omega_E^\tau\backslash Re\,\Omega_E^\tau$, $H^{appr}_{E,\tau}$ is found using (2.25) (as an appropriate $\bar\partial$- equation for H on $\Omega_E^\tau\backslash Re\,\Omega_E^\tau$), the property that

H_+ and H_- are the boundary values of H in the limits on $Re\,\Omega_E^\tau$ from $\Omega_E^\tau\backslash Re\,\Omega_E^\tau$ (see (2.11), (2.31a)) and estimate (2.29), more precisely, $H_{E,\tau}^{appr}$ is found from equation (2.35) discussed below;

$$\begin{aligned}&\hat v^{\pm}_{appr}(p,\tau,E) \overset{(2.28)}{=} \lim_{(k,p)\in\Omega_E^{\tau,\pm},|k|\to\infty} H_{E,\tau}^{appr}(k,p) \ \text{ for almost each } \ p\in\mathcal{B}_{2\tau\sqrt{E}},\\ &\Omega_E^{\tau,\pm}=\{(k,p)\in\Omega_E^\tau:\ \frac{Im\,k}{|Im\,k|}=\pm\frac{p\times Re\,k}{|p\times Re\,k|},\ Im\,k\neq 0, p\neq 0\},\end{aligned} \tag{2.31c}$$

see also (2.55).

To present (2.31b) in more detail we emphasize that (2.25) is only an approximate $\bar\partial$-equation for H on $\Omega_E^\tau\backslash Re\,\Omega_E^\tau$, $\tau\in]0,1[$:

$$\begin{aligned}&\partial_{\bar k}\chi_{2\tau\sqrt{E}}H(k,p)=-2\pi\int\limits_{\mathbb{R}^3}(\sum_{j=1}^{3}\xi_j d\bar k_j)\chi_{2\tau\sqrt{E}}H(k,-\xi)\chi_{2\tau\sqrt{E}}H(k+\xi,p+\xi)\times\\ &\delta(\xi^2+2k\xi)d\xi+R_{E,\tau}(k,p)\ \text{ for }\ \chi_{2\tau\sqrt{E}}H\ \text{ on }\ \Omega_E^\tau\backslash Re\,\Omega_E^\tau,\ \tau\in]0,1[,\end{aligned} \tag{2.32}$$

where χ_r denotes the multiplication operator by the function

$$\chi_r(p)=1\ \text{ for }\ |p|<r,\ \chi_r(p)=0\ \text{ for }\ |p|\geq r,$$

where $p\in\mathbb{R}^3$, $r>0$, and, as a corollary,

$$\begin{aligned}&\chi_{2\tau\sqrt{E}}H=H\ \text{ on }\ \Omega_E^\tau\backslash Re\,\Omega_E^\tau,\\ &\chi_{2\tau\sqrt{E}}H\equiv 0\ \text{ on }\ (\Omega_E\backslash Re\,\Omega_E)\backslash\Omega_E^\tau,\end{aligned} \tag{2.33}$$

$R_{E,\tau}$ is a remainder which can be written explicitly (see [No8]). Proceeding from (2.32), the definition of $H_\pm$ of (2.11), (2.31a) and estimate (2.29), we obtain for $H_{E,\tau}=H\big|_{\Omega_E^\tau\backslash Re\,\Omega_E^\tau}$ a nonlinear integral equation of the form (see [No8]):

$$H_{E,\tau}=H_{E,\tau}^0+M_{E,\tau}(H_{E,\tau})+Q_{E,\tau},\quad \tau\in]0,1[, \tag{2.34}$$

where $H_{E,\tau}^0$ is expressed explicitly in terms of $H_\pm$, $M_{E,\tau}$ is a nonlinear integral operator and $Q_{E,\tau}$ is a remainder expressed in terms of $R_{E,\tau}$ and admitting an estimate using (2.29). Proceeding from (2.34) we define $H_{E,\tau}^{appr}$ of (2.31b) by the nonlinear integral equation

$$H_{E,\tau}^{appr}=H_{E,\tau}^0+M_{E,\tau}(H_{E,\tau}^{appr}),\quad \tau\in]0,1[. \tag{2.35}$$

2.4. Precise formulas for $H_{E,\tau}^0$ and $M_{E,\tau}$. Precise formulas for $H_{E,\tau}^0$ and $M_{E,\tau}$ of (2.35) (see formulas (2.49), (2.50) given below) are based on the following additional geometric considerations. One can see that

$$(k,p)\in\Omega_E\Longleftrightarrow k\in\Sigma_{E,p}^{(3)}=\{k\in\mathbb{C}^3:\ k^2=E,\ p^2=2kp\} \tag{2.36}$$

for fixed $E > 0$ and $p \in \mathbb{R}^3$. In addition,

$$\Sigma^{(3)}_{E,p} \approx \Sigma^{(2)}_{E-p^2/4}, \quad \Sigma^{(2)}_{E} = \{k \in \mathbb{C}^2 : \ k^2 = E\}, \quad \text{for} \ \ 0 < |p| < 2\sqrt{E}. \tag{2.37}$$

It is interesting to note that the inverse scattering theory at fixed energy $E > 0$ in dimension $d = 2$ mentioned in Remark 1.5 of Introduction is, actually, based on some complex analysis on $\Sigma^{(2)}_{E}$. In particular, it is very much used in these 2D considerations that the following formulas

$$\lambda = \frac{k_1 + ik_2}{E^{1/2}}, \ k_1 = \big(\lambda + \frac{1}{\lambda}\big), \ k_2 = \big(\frac{1}{\lambda} - \lambda\big)\frac{iE^{1/2}}{2} \tag{2.38}$$

give a diffeomorphism between $\Sigma^{(2)}_{E}$, $E > 0$, and $\mathbb{C}\backslash 0$ (with the variable λ). Proceeding from these observations we introduce some convenient coordinates on Ω^{τ}_{E}, $E > 0$, $\tau \in]0, 1]$, as follows ([No8]).

Let

$$\mathcal{L}_{\nu} = \{p \in \mathbb{R}^3 : \ p = t\nu, \ t \in \mathbb{R}\}, \tag{2.39a}$$

$$\mathcal{B}_{r,\nu} = \{p \in \mathcal{B}_r : \ p \notin \mathcal{L}_{\nu}\}, \tag{2.39b}$$

$$\Omega^{\tau}_{E,\nu} = \{(k, p) \in \Omega^{\tau}_{E,\nu} : \ p \notin \mathcal{L}_{\nu}\}, \quad Re\, \Omega^{\tau}_{E,\nu} = \{(k, p) \in Re\, \Omega^{\tau}_{E,\nu} : \ p \notin \mathcal{L}_{\nu}\}, \tag{2.39c}$$

where $\nu \in \mathbb{S}^2$, $r > 0$, $E > 0$, $\tau \in]0, 1]$. For $p \in \mathbb{R}^3\backslash\mathcal{L}_{\nu}$ we consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned} &\theta(p), \ \omega(p) \ \ \text{smoothly depend on} \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_{\nu} \\ &\text{take their values in} \ \ \mathbb{S}^2 \ \ \text{and} \\ &\theta(p)p = 0, \ \ \omega(p)p = 0, \ \ \theta(p)\omega(p) = 0. \end{aligned} \tag{2.40}$$

Conditions (2.40) imply that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \ \ \text{for} \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_{\nu} \tag{2.41a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \ \ \text{for} \ \ p \in \mathbb{R}^3\backslash\mathcal{L}_{\nu}, \tag{2.41b}$$

where $\times$ denotes vector product. To satisfy (2.40), (2.41a) one can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \ \omega(p) = \frac{p \times \theta(p)}{|p|}, \ p \in \mathbb{R}^3\backslash\mathcal{L}_{\nu}. \tag{2.42}$$

Let $E > 0$, $\tau \in]0, 1]$, $\nu \in \mathbb{S}^2$ and the functions θ and ω of (2.40), (2.41a) be fixed. Then ([No8]) the following formulas give a diffeomorphism between $\Omega^{\tau}_{E,\nu}$ and $(\mathbb{C}\backslash 0) \times \mathcal{B}_{2\tau\sqrt{E},\nu}$:

$$(k, p) \to (\lambda, p), \quad \text{where} \ \ \lambda = \lambda(k, p) = \frac{k(\theta(p) + i\omega(p))}{(E - p^2/4)^{1/2}}, \tag{2.43}$$

$$\begin{aligned}&(\lambda,p)\to(k,p),\quad \text{where}\ \ k=k(\lambda,p,E)=\kappa_1(\lambda,p,E)\theta(p)+\kappa_2(\lambda,p,E)\omega(p)+p/2,\\ &\kappa_1(\lambda,p,E)=\big(\lambda+\frac{1}{\lambda}\big)\frac{(E-p^2/4)^{1/2}}{2},\ \kappa_2(\lambda,p,E)=\big(\frac{1}{\lambda}-\lambda\big)\frac{i(E-p^2/4)^{1/2}}{2},\end{aligned}\tag{2.44}$$

where $(k,p)\in\Omega^\tau_{E,\nu}$, $(\lambda,p)\in(\mathbb{C}\backslash 0)\times\mathcal{B}_{2\tau\sqrt{E},\nu}$; in addition, formulas (2.43), (2.44) give also diffeomorphisms between $Re\,\Omega^\tau_{E,\nu}$ and $\mathcal{T}\times\mathcal{B}_{2\tau\sqrt{E},\nu}$ and between $\Omega^\tau_{E,\nu}\backslash Re\,\Omega^\tau_{E,\nu}$ and $(\mathbb{C}\backslash(0\cup\mathcal{T}))\times\mathcal{B}_{2\tau\sqrt{E},\nu}$, where

$$\mathcal{T}=\{\lambda\in\mathbb{C}:\ |\lambda|=1\}.\tag{2.45}$$

We consider λ, p of (2.43), (2.44) as coordinates on $\Omega^\tau_{E,\nu}$ and on Ω^τ_E, $E>0$, $\tau\in]0,1]$.

One can see a considerable similarity between formulas (2.43), (2.44) and formulas (2.38).

In the coordinates λ, p the restriction $H_{E,\tau}=H\big|_{\Omega^\tau_E\backslash Re\,\Omega^\tau_E}$ can be written as

$$H_{E,\tau}=H(\lambda,p,E)=H(k(\lambda,p,E),p),\ \lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0),\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},\tag{2.46a}$$

and $H_\pm$ of (2.31a) can be written as

$$H_\pm(\lambda,p,E)=H_\pm(k(\lambda,p,E),p),\ \lambda\in\mathcal{T},\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu}.\tag{2.46b}$$

In addition,

(a) the following limit relation is valid:

$$H_\pm(\lambda,p,E)=H(\lambda(1\mp 0),p,E),\ \lambda\in\mathcal{T},\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu};\tag{2.47}$$

(b) the $\bar\partial$-equation (2.32) takes the form

$$\frac{\partial}{\partial\bar\lambda}H(\lambda,p,E)=(H,H)_{E,\tau}(\lambda,p)+R_{E,\tau}(\lambda,p),\ \lambda\in\mathbb{C}\backslash(\mathcal{T}\cup 0),\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},\tag{2.48}$$

where (H,H) is defined by (2.51) and $R_{E,\tau}(\lambda,p)=R_{E,\tau}(k(\lambda,p,E),p)$;

(c) the following formulas for $H^0_{E,\tau}$ and $M_{E,\tau}$ of (2.35) hold:

$$\begin{aligned}&H^0_{E,\tau}(\lambda,p)=\frac{1}{2\pi i}\int\limits_{\mathcal{T}}H_+(\zeta,p,E)\frac{d\zeta}{\zeta-\lambda},\ \lambda\in\mathcal{D}_+,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},\\ &H^0_{E,\tau}(\lambda,p)=-\frac{1}{2\pi i}\int\limits_{\mathcal{T}}H_-(\zeta,p,E)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)},\ \lambda\in\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},\end{aligned}\tag{2.49}$$

$$\begin{aligned}&M_{E,\tau}(U)(\lambda,p)=M^+_{E,\tau}(U)(\lambda,p)=\\ &-\frac{1}{\pi}\iint\limits_{\mathcal{D}_+}(U,U)_{E,\tau}(\zeta,p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta-\lambda},\ \lambda\in\mathcal{D}_+,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},\end{aligned}\tag{2.50a}$$

$$M_{E,\tau}(U)(\lambda,p)=M^-_{E,\tau}(U)(\lambda,p)=$$
$$-\frac{1}{\pi}\iint\limits_{\mathcal{D}_-}(U,U)_{E,\tau}(\zeta,p)\frac{\lambda dRe\,\zeta dIm\,\zeta}{\zeta(\zeta-\lambda)},\ \lambda\in\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu}, \tag{2.50b}$$

$$(U_1,U_2)_{E,\tau}(\lambda,p)=$$
$$-\frac{\pi}{4}\int\limits_{-\pi}^{\pi}\left((E-p^2/4)^{1/2}\frac{sgn(|\lambda|^2-1)(|\lambda|^2+1)}{\bar\lambda|\lambda|}(\cos\varphi-1)-|p|\frac{1}{\bar\lambda}\sin\varphi\right)\times$$
$$U_1(z_1(\lambda,p,E,\varphi),-\xi(\lambda,p,E,\varphi))U_2(z_2(\lambda,p,E,\varphi),p+\xi(\lambda,p,E,\varphi))\times$$
$$\chi_{2\tau\sqrt{E}}(\xi(\lambda,p,E,\varphi))\chi_{2\tau\sqrt{E}}(p+\xi(\lambda,p,E,\varphi))d\varphi,\ \lambda\in\mathcal{D}_+\cup\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu}, \tag{2.51}$$

where
$$\mathcal{D}_+=\{\lambda\in\mathbb{C}:\ 0<|\lambda|<1\},\ \mathcal{D}_-=\{\lambda\in\mathbb{C}:\ |\lambda|>1\} \tag{2.52}$$

U_1,U_2,U_3 are test functions on $(\mathcal{D}_+\cup\mathcal{D}_-)\times\mathcal{B}_{2\tau\sqrt{E},\nu}$,

$$z_1(\lambda,p,E,\varphi)=\frac{k(\lambda,p,E)(\theta(-\xi(\lambda,p,E,\varphi))+i\omega(-\xi(\lambda,p,E,\varphi)))}{(E-|\xi(\lambda,p,E,\varphi)|^2/4)^{1/2}},$$
$$z_2(\lambda,p,E,\varphi)=\frac{(k(\lambda,p,E)+\xi(\lambda,p,E,\varphi))(\theta(p+\xi(\lambda,p,E,\varphi))+i\omega(p+\xi(\lambda,p,E,\varphi)))}{(E-|p+\xi(\lambda,p,E,\varphi)|^2/4)^{1/2}}, \tag{2.53}$$

$$\xi(\lambda,p,E,\varphi)=Re\,k(\lambda,p,E)(\cos\varphi-1)+k^\perp(\lambda,p,E)\sin\varphi,$$
$$k^\perp(\lambda,p,E,\varphi)=\frac{Im\,k(\lambda,p,E)\times Re\,k(\lambda,p,E)}{|Im\,k(\lambda,p,E)|}, \tag{2.54}$$

where $k(\lambda,p,E)$ is defined by (2.44) and θ, ω are the functions of (2.40), (2.41a).

Note also that in the coordinates λ, p formula (2.31c) takes the form

$$\hat v^+_{appr}(p,\tau,E)=\lim_{\lambda\to 0}H^{appr}_{E,\tau}(\lambda,p),\ \lambda\in\mathcal{D}_+,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu},$$
$$\hat v^-_{appr}(p,\tau,E)=\lim_{\lambda\to\infty}H^{appr}_{E,\tau}(\lambda,p),\ \lambda\in\mathcal{D}_-,\ p\in\mathcal{B}_{2\tau\sqrt{E},\nu}. \tag{2.55}$$

This completes a formal description (without estimates) of equations and formulas involved in (2.3), (2.31).

3. Error and stability estimates of [No8]

Note that it is more convenient (for estimates) to consider (2.3), (2.31) under condition (2.18), $\mu\ge 2$, than under our initial condition (1.7), $n>d=3$. We have, in particular, the following results of [No8].

Let $\hat v$ satisfy (2.18), where $\mu\ge 2$ and $\|\hat v\|_{\alpha,\mu}\le D$ for some $D>0$. Let f be the scattering amplitude for equation (1.1). Let

$$0<\tau<\tau_1(\alpha,\beta,\mu,\mu_0,\sigma,D),\ E\ge E_1(\alpha,\beta,\mu,\mu_0,\sigma,D,g_1,g_2) \tag{3.1}$$

for some special τ_1 and E_1 (which can be given explicitly, see [No8]), where β, μ, μ_0, σ, D, g_1, g_2 are some additional fixed numbers such that

$$\begin{aligned}&2 \le \mu_0 \le \mu, \quad 0 < \sigma < 1, \quad 0 < \beta < \min(\alpha, \sigma, 1/2),\\ &g_1 > 1, \quad g_2 > Const_2(\mu), \quad g_2 > g_1.\end{aligned} \tag{3.2}$$

In addition, in particular,

$$\begin{aligned}&E_1 \to +\infty \ \text{ as } \ D \to +\infty,\\ &0 < \tau_1 < 1 \ \text{ and (in the framework of } \ [No8])\ \tau_1 \to 0, \ \text{ as } \ D \to +\infty.\end{aligned} \tag{3.3}$$

Let also λ, p be the coordinates of (2.43), (2.44). Then ([No8]):

I. The following estimates hold:

$$\|f\|_{C(\mathcal{M}_E),\mu} < g_1 D, \quad \|f\|_{C^\alpha(\mathcal{M}_E),\mu} < g_2 D, \tag{3.4}$$

$$\begin{aligned}&|H^0_{E,\tau}(\lambda,p)| \le R(\alpha,\beta,\mu,\sigma,D,E,|p|) \overset{\text{def}}{=} \left(2^{\mu/2}D + \frac{Const_3(\alpha,\beta,\mu,\sigma)D^2}{E^{\beta/2}}\right)\frac{1}{(1+|p|)^\mu},\\ &\lambda \in \mathcal{D}_+ \cup \mathcal{D}_-, \quad p \in \mathcal{B}_{2\tau\sqrt{E},\nu},\end{aligned} \tag{3.5}$$

where $H^0_{E,\tau}$ is the function of (2.35), (2.49).

II. All equations of (2.3), (2.31) (equations (2.20), (2.35)) for finding $\hat v^\pm_{appr}(\cdot,\tau,E)$ on $\mathcal{B}_{2\tau\sqrt{E}}$ from f on $\mathcal{M}_E$ are uniquely solvable (by successive approximations) and the following error estimate holds:

$$|\hat v(p) - \hat v^\pm_{appr}(p,\tau,E)| \le \frac{Const_4(\mu,\mu_0,\tau)D^2}{(1+|p|)^{\mu_0}(1+2\tau\sqrt{E})^{\mu-\mu_0}}, \ p \in \mathcal{B}_{2\tau\sqrt{E},\nu}. \tag{3.6}$$

III. In addition, if $\tilde f$ is an approximation to f, where estimates (3.4) are valid also for $\tilde f$ in place of f, then by means of (2.31), (2.35) $\tilde f$ on $\mathcal{M}_E$ determines $\tilde{\hat v}^\pm_{appr}$ on $\mathcal{B}_{2\tau\sqrt{E}}$ and the following estimate holds:

$$\begin{aligned}&|\hat v^\pm_{appr}(p,\tau,E) - \tilde{\hat v}^\pm_{appr}(p,\tau,E)| \le\\ &Const_5\frac{(1+\ln E)\|f-\tilde f\|_{C(\mathcal{M}_E),\mu}}{(1+|p|)^{\mu_0}} + Const_6(\varepsilon\beta)\times\\ &\frac{(Const_{7,1}(\mu)g_2D + Const_{7,2}(\beta,\mu)(g_1D)^2E^{-\beta/2})(\|f-\tilde f\|_{C(\mathcal{M}_E),\mu})^{1-\varepsilon}}{(1+|p|)^{\mu_0}},\\ &p \in \mathcal{B}_{2\tau\sqrt{E},\nu}, \ \varepsilon \in]0,1[.\end{aligned} \tag{3.7}$$

More precisely, $\tilde{\hat v}^\pm_{appr}$ on $\mathcal{B}_{2\tau\sqrt{E}}$ is determined from f on $\mathcal{M}_E$ via (2.31), (2.35), (2.49), (2.50), (2.55) with $\tilde f$, $\tilde H_\gamma$, $\tilde H_\pm$, $\tilde H^0_{E,\tau}$, $\tilde H^{appr}_{E,\tau}$, $\tilde{\hat v}^\pm_{appr}$ in place of f, H_γ, $H_\pm$, $H^0_{E,\tau}$, $H^{appr}_{E,\tau}$,

$\hat{v}^{\pm}_{appr}$, where $\tilde{H}^0_{E,\tau}$ arising in (2.49) is corrected also as follows:

$$\begin{aligned}
&\tilde{H}^0_{E,\tau}(\lambda,p) \to \tilde{H}^0_{E,\tau}(\lambda,p)\\
&\text{if}\ \ |\tilde{H}^0_{E,\tau}(\lambda,p)| \le R(\alpha,\beta,\mu,\sigma,D,E,|p|),\\
&\tilde{H}^0_{E,\tau}(\lambda,p) \to R(\alpha,\beta,\mu,\sigma,D,E,|p|)\frac{\tilde{H}^0_{E,\tau}(\lambda,p)}{|\tilde{H}^0_{E,\tau}(\lambda,p)|}\\
&\text{if}\ \ |\tilde{H}^0_{E,\tau}(\lambda,p)| > R(\alpha,\beta,\mu,\sigma,D,E,|p|),
\end{aligned} \tag{3.8}$$

where R is radius of (3.5).

In connection with (3.5), (3.8) note also that if $\tilde{f} = f$ (that is the scattering amplitude is given with no errors) and $\tilde{H}^0_{E,\tau}$ is calculated from $\tilde{f}$ with no errors, then $\tilde{H}^0_{E,\tau} = H^0_{E,\tau}$ and (due to (3.5)) correction (3.8) is not necessary. However, if some of these errors are present, then (3.8) seems to be necessary, in general, for solvability of (2.35) by iterations and for validity of (3.7).

The remark that (2.17) follows from (1.7), $d = 3$, and estimates (3.6), (3.7) imply the estimates of (2.3), (2.4).

4. Results of [ABR3]

The monochromatic inverse scattering method of [No8] was implemented for the first time numerically in [ABR2], [ABR3] for the case of the inverse scattering problem for the acoustic equation

$$-\Delta\psi = \big(\frac{\omega}{c(x)} + i\alpha(x,\omega)\big)^2\psi, \quad x \in \mathbb{R}^3, \tag{4.1}$$

with velocity of sound $c(x)$, amplitude absorption coefficient $\alpha(x,\omega)$, at fixed frequency ω, under the assumption that

$$c(x) \equiv c_0,\ \ \alpha(x,\omega) \equiv 0 \ \ \text{for}\ \ |x| \ge r. \tag{4.2}$$

The possibility of applications of the method of [No8] (as well as of any method for solving Problem 1.1 for equation (1.1) under assumption (1.2)) to the inverse scattering problem for equation (4.1) under assumptions (4.2) (with known background c_0) is based on the fact that (4.1) can be written in the form (1.1), where

$$v = \frac{\omega^2}{c_0^2} - \big(\frac{\omega}{c(x)} + i\alpha(x,\omega)\big)^2, \quad E = \frac{\omega^2}{c_0^2}, \tag{4.3a}$$

and that, as a corollary of (4.2),

$$v = v(x,\omega) \equiv 0 \ \ \text{for}\ \ |x| \ge r. \tag{4.3b}$$

The main (or, at least, the most difficult for numerical implementation) points of monochromatic inverse scattering method of [No8] consists in finding h_γ from f via (2.20) and finding $H^{appr}_{E,\tau}$ from $H^0_{E,\tau}$ via (2.35). One can see that (2.20) is a (standard) linear Fredholm integral equation of the second type, whereas (2.35) is a nonlinear integral equation. In [ABR3] equation (2.35) is solved for $\tau = 1$ by iterations organized as follows:

$$\begin{aligned}&\tilde{H}^j_E(\lambda,p) = W_E(|p|)\big(H^0_E(\lambda,p) + M_E(\tilde{\tilde{H}}_E^{j-1})(\lambda,p)\big),\\ &\tilde{\tilde{H}}_E^{j=0}(\lambda,p) = W_E(|p|)H^0_E(\lambda,p),\ \tilde{\tilde{H}}_E^{j} = (1-\varepsilon)\tilde{\tilde{H}}_E^{j-1} + \varepsilon\tilde{H}^j_E,\ 0<\varepsilon\le 1,\end{aligned} \tag{4.4}$$

where ε is a relaxation parameter and $W_E(|p|)$ is a sufficiently regular filter such that

$$\begin{aligned}&W_E(r) = 1 \ \text{ for } \ 0 \le r \le 2\tau_0\sqrt{E},\ W_E(r) = 0 \ \text{ for } \ r \ge 2\tau_1\sqrt{E}\\ &\text{for some } \ \tau_0 \ \text{ and } \ \tau_1,\ \ 0<\tau_0<\tau_1\le 1.\end{aligned}$$

In addition, the numerical implementation of [ABR3] is reduced to the algorithm of [ABR1] if (2.35) is "solved" by the zero approximation:

$$H^{appr}_{E,1} \approx H^0_{E,1}. \tag{4.5}$$

Note that sufficiently strong scatterers are successfully reconstructed in [ABR3] by the method of [No8] in the framework of numerical simulations for the acoustic equation (4.1). We emphasize that the Born approximation method does not give already a satisfactory reconstruction result for these scatterers. In [ABR3] it is also numerically shown that the method of [No8] is considerably more precise than its approximate version solving (2.35) by the zero approximation and actually used in [ABR1].

Note that in the numerical simulations on [ABR3] is is assumed that c and α of (4.1) are spherically symmetric functions of x. However, this condition is used in [ABR3] for reducing the volume of numerical operations only and is not an assumption of the method.

Note that some of results of [ABR3] are presented already in [ABR2].

5. Open problem for the acoustic case

Note that the potential $v = v(x,\omega)$ of (4.3) depends on the frequecy ω in constrast with the energy independent potential $v = v(x)$ of the Schrödinger equation (1.1).

For the acoustic equation (4.1) under condition (4.2) and even with $\alpha \equiv 0$ on the whole space (in dimension $d \ge 2$), analogs of (2.2), (2.4) are not obtained yet, in general, in the framework of (approximate but) sufficiently stable monochromatic inverse scattering (by methods of [No6]-[No8] or by other methods). This is an important open problem for the acoustic case. In this connection, in addition to results of [No6]-[No8] and [BBMRS], [BMR], [ABR3], we would like to mention also that:

In the particular case of the 3-dimensional acoustic equation in a half-space with horizontally-homogeneous velocity c (and zero absorption α) a stable monochromatic inverse scattering method with the error estimate (for $\omega \to +\infty$) of the type (2.2) was developed in [MNPSF], [HeNo].

An interesting stability analysis for monochromatic acoustical inverse scattering with a decreasing error term as $\omega \to +\infty$ is developed in [P] for $d = 2$.

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